# Orientability and The Poincaré Duality theorem

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## Abstract

In this poster session, I will explain duality relationship between de Rham cohomology and de Rham cohomology with compact support of a smooth manifold, entitled as *Poincaré duality*. I will explain definitions of de Rham complex and twisted de Rham complex, Mayer-Vietoris sequence, applications.

## Dual space and five lemma

1. For a given real vector space $V$ there is an associated real vector space $Hom_{\mathbb{R}}(V, \mathbb{R})$ of $\mathbb{R}$-linear functionals, known as dual space of $V$ and denoted by $V^*$.

**Theorem 1 (Five lemma)** *A lemma, which states that for a given commutative diagram of additive abelian groups with exact rows, if $f_0$, $f_1$, $f_3$ and $f_4$ are isomorphisms then $f_2$ is an isomorphism.*

$$\begin{array}{ccccccccccc} \cdots & \text{---} & A & \text{---} & B & \text{---} & C & \text{---} & D & \text{---} & E & \text{---} & \cdots \\ & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \\ \cdots & \text{---} & a & \text{---} & b & \text{---} & c & \text{---} & d & \text{---} & e & \text{---} & \cdots \end{array}$$

## Oriented smooth manifold, good cover and de Rham cohomology

1. The collection of $C^\infty$-differential forms on $\mathbb{R}^n$ together with de Rham operator $d : \Omega^p(\mathbb{R}^n) \to \Omega^{p+1}(\mathbb{R}^n)(1 \le p \le n-1)$ is called **de Rham complex**. The quotient $ker(d)/img(d)$ is the **$p$-th de Rham cohomology group,** $H^p(\mathbb{R}^n)$, similarly we have, **de Rham complex with compact support** and the cohomology group $H^p_c(\mathbb{R}^n)$.
2. A connected, Hausdorff, second countable and locally euclidean topological space with smooth atlas is called **smooth manifold**. If Jacobian of transition maps remain of same sign, smooth manifold is called **oriented smooth manifold** otherwise **non-oreinted smooth manifold**.
3. An open cover $\{U_\alpha\}_{\alpha \in I}$ of a manifold $M$ is called good cover if each non-empty finite intersection $U_{\alpha_0} \cap \ldots U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. For a finite index set, good cover is said to be of finite type.
4. Poincaré has computed $H^p(\mathbb{R}^n)$, $H^p_c(\mathbb{R}^n)$ entitled as **Poincaré Lemma**.
5. The **Mayer-Vietoris sequence** $0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$ is exact and useful to compute $H^*(M)$, where $M = U \cup V$.
6. Similarly, we have Mayer-Vietoris sequence $0 \to \Omega^*_c(U \cap V) \to \Omega^*_c(U) \oplus \Omega^*_c(V) \to \Omega^*_c(M) \to 0$ for compact supports, which is useful to compute $H^*_c(M)$.
7. Using Mayer-Vietoris sequence, induction principle and Poincaré lemma, if a manifold has finite good cover then its cohomology as wells as cohomology with compact support are finite dimensional.

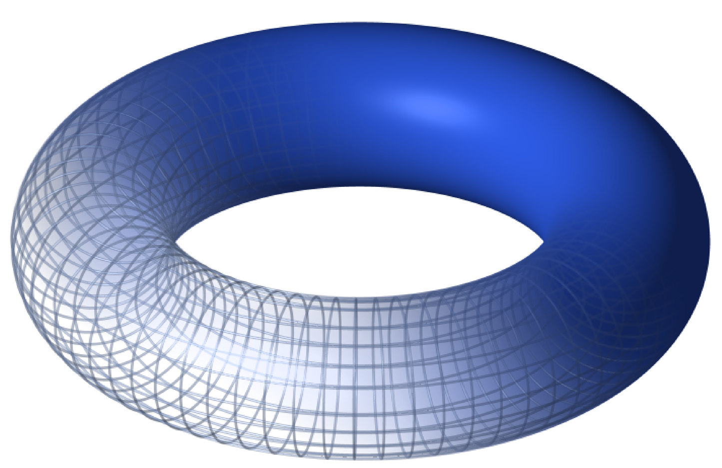

Figure: smooth manifold

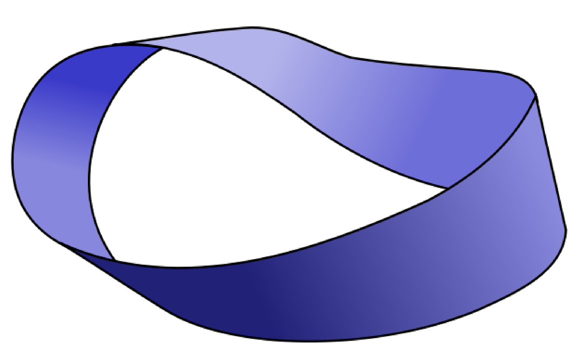

Figure: Mobius Strip

## Poincaré duality(orientable)

1. Let $V$ and $W$ be finite-dimensional vector spaces. The pairing $\langle , \rangle : V \otimes W \to \mathbb{R}$ is nondegenerate iff the map $v \mapsto \langle v, \rangle$ defines an isomorphism $V \xrightarrow{\sim} W^*$ for some fixed $v \in V$ and the map $w \mapsto \langle , w \rangle$ defines an isomorphism $W \xrightarrow{\sim} V^*$ for some fixed $w \in W$.
2. For oriented smooth manifold the wedge product and integration(using Stokes' theorem) of smooth forms descends to cohomology.

**Theorem 2** *For an oriented smooth manifold $M$ with finite good cover, there is a non-degenerate pairing, $H^q(M) \otimes H^{n-q}_c(M) \to \mathbb{R}$ given by $(\tau, \mu \mapsto \int \tau \wedge \mu)$ equivalently, $H^q(M) \simeq (H^{n-q}_c(M))^*$*

**Proof** The proof can be described as follows. ;

1. Let $M = \overset{l}{\underset{i=1}{\cup}} U_i$ be finite good cover.
2. Induced long exact cohomology sequences from 5 and 6 can be paired(using non-degenerate pairing) to get square diagrams with both exact rows as follows.

$$\begin{array}{ccccc} ..H^q((U_0 \cup ..U_{l-1}) \cap U_l) & \longrightarrow & H^{q+1}(U_0 \cup ..U_{l-1} \cup U_l) & \longrightarrow & H^{q+1}(U_0 \cup ..U_{l-1}) \oplus H^{q+1}(U_l).. \\ \wr\!\mid & & \downarrow f_2 & & \wr\!\mid \\ ..(H^{n-q}_c((U_0 \cup ..U_{l-1}) \cap U_l))^* & \to & (H^{n-q-1}_c(U_0 \cup ..U_{l-1} \cup U_l))^* & \to & (H^{n-q-1}(U_0 \cup ..U_{l-1}))^* \oplus (H^{n-q-1}(U_l))^*.. \end{array}$$

3. Using five lemma and Poincaré lemma, Poincaré duality holds.
4. The finiteness condition on good cover is not necessary[5, Page no. 14 and 198].

## Twisted de Rham complex, cohomology and orientation bundle

1. For a given vector bundle $E$ on $M$, we can define the space of $E$-valued smooth $q$-forms to be global sections of vector bundle $\wedge^q T^*M \otimes E$. There is an $\mathbb{R}$-algebra $\Omega^*(M, E)$.
2. For a flat vector bundle $E$ with trivialization $\phi = \{U_\alpha, e_\alpha\}_{\alpha \in I}$, we can define a differential operator $d_E : \Omega^p(M, E) \to \Omega^{p+1}(M, E)$, locally given by $d_E(\sum_i \omega_i \otimes e_{\alpha_i}) = \sum_i d(\omega_i) \otimes e_{\alpha_i}$ and we have differential complex $\Omega^*_\phi(M, E)$ along with cohomology $H^*_\phi(M, E)$ depending on trivialization.

**Proposition 1** *For given two trivialization $\phi$ and $\psi$ with associated cocycle maps $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ and same open cover $\{U_\alpha\}_{\alpha \in I}$. If there exists locally constant functions $\lambda : U_\alpha \to GL_n(\mathbb{R})$ such that $g_{\alpha\beta} = \lambda_\alpha h_{\alpha\beta} \lambda_\beta^{-1}$, there are isomorphisms $H^*_\phi(M, E) \simeq H^*_\psi(M, E)$.*

1. For a manifold $M$, atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$, transition map $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$, **orientation line bundle** $L$ is a line bundle with co-cycle map,

$$g^L_{\alpha\beta} = \begin{cases} 1, & \text{if } J(g_{\alpha\beta} > 0 \\ 0, & \text{if } J(g_{\alpha\beta} = 0 \\ -1, & \text{if } J(g_{\alpha\beta} < 0 \end{cases}$$

**Proposition 2** *For two trivializations $\phi'$ and $\psi'$ of $L$ induced from two atlases $\phi$ and $\psi$ on smooth manifold $M$, then the twisted complexes $\Omega^*_{\phi'}(M, L)$, $\Omega^*_{\psi'}(M, L)$ are isomorphic also cohomologies $H^*_{\phi'}(M, L)$, $H^*_{\psi'}(M, L)$ are.*

## Poincaré duality(non-orientable)

1. Define the **twisted de Rham complex** $\Omega^*(M, L)$ and **twisted de Rham cohomology** $H^*(M, L)$ to be $\Omega^*_{\phi'}(M, L)$ and $H^*_{\phi'}(M, L)$ for any trivialization $\phi'$ of $L$ induced from $M$. Similarly, we can have **twisted de Rham cohomology with compact support,** $H^*_c(M, L)$.
2. If a trivialization $\psi$ on $L$ is not induced from $M$ then $H^*_\psi(M, L)$ may not be equal to $H^*(M, L)$.
3. A **density** on $M$(dimension $n$) is an element of $\Omega^n(M, L)$, equivalently a section of the bundle $(\wedge^n T^*M \otimes L)$.
4. The transition function for the bundle $\wedge^n T^*M \otimes L$ is $\frac{1}{|J(g_{\alpha\beta})|}$; and global integration of density is defined.
5. Similar to the case of orientable manifold, wedge product and integration descends to cohomology group.

**Theorem 3** *On a manifold $M$ of dimension $n$ with a finite good cover, there are nondegenerate pairings $H^q(M) \otimes_{\mathbb{R}} H^{n-q}_c(M, L) \to \mathbb{R}$ and $H^q_c(M) \otimes_{\mathbb{R}} H^{n-q}(M, L) \to \mathbb{R}$ equivalently, $H^q(M) \simeq (H^{n-q}_c(M, L))^*$.*

## Applications

1. Let $M$ be a connected manifold of dimension $n$ having a finite good cover. Then,[1, Corollary 7.8.1]

$$H^n(M) = \begin{cases} \mathbb{R}, & \text{if } M \text{ is compact orientable} \\ 0, & \text{otherwise} \end{cases}$$

2. Using universal coefficient theorem and Poincaré duality we can deduce, a closed manifold of odd dimension has Euler characteristic 0[2, Corollary 3.37].

## References

[1] Raoul Bott, Loring W Tu, et al.; *Differential forms in algebraic topology*, volume 82. Springer, 1982

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[5] Halperin S., Greub W. and Vanstone R.; *Connections, Curvature and Cohomology, vol. 1*, Academic Press, New York and London, 1972.