On d-holomorphic connections

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Introduction

1. In 1957, Michael Francis Atiyah proposed a criterion for existence of holomorphic connection on holomorphic bundle on compact Riemann surface.[3]
2. In 1882, Felix Klein had introduced Klein surface as topological 2-manifold having atlas along with transition map, which is either holomorphic or anti-holomorphic [5] and Norman Alling and Newcomb Greenleaf studied d-holomorphic functions on the Klein surface[1].

Objectives

The aim is

1. to define d-holomorphic connection as covariant differential operator.
2. to describe criterion for existence of d-holomorphic connection on d-holomorphic bundle on Klein surface.

Definitions

1. A d-holomorphic vector bundle E(rank r)[6] on $(X_d, \mathfrak{X}_d)$ is a quotient space $E = \coprod_{\{(U_i,z_i)\}_{i\in I}} U_i \times \mathbb{C}^r /_{\sim}$ with cocycle map g_{ij}^E(either holomorphic or antiholomorphic) such that,

$$g_{ki}^E = \left\{ \begin{matrix} g_{kj}^E \circ g_{ji}^E \\ g_{kj}^E \circ \overline{g}_{ji}^E \end{matrix} \right\}, \quad g_{ij}^E = \begin{cases} (g_{ji}^E)^{-1}, \text{if } z_j \circ z_i{}^{-1} \text{ is ana.} \\ (\overline{g}_{ji}^E)^{-1}, \text{if } z_j \circ z_i{}^{-1} \text{ is antiana.} \end{cases}$$

   and the equivalence relation $\sim$ is defined as follows.
   For $(x_i, \xi_i) \in U_i \times \mathbb{C}^r$ and $(x_j, \xi_j) \in U_j \times \mathbb{C}^r$ then $(x_i, \xi_i) \sim (x_j, \xi_j)$ if,

$$x_i = x_j \text{ and } \xi_j = \begin{cases} g_{ji}^E(x_i)\xi_i, \text{if } z_j \circ z_i{}^{-1} \text{ is holomorphic} \\ g_{ji}^E(x_i)\overline{\xi}_i, \text{if } z_j \circ z_i{}^{-1} \text{ is antiholomorphic} \end{cases}$$

2. A maximal family(independent from atlas) $\{f_{U_i}\}_{i\in I}$ of holomorphic functions w.r.t an atlas $\{(U_i, z_i)\}_{i\in I}$ such that,

$$f_{U_j} = \begin{cases} f_{U_i}, \text{ if } z_j \circ z_i^{-1} \text{ is analytic} \\ \overline{f}_{U_i}, \text{ if } z_j \circ z_i^{-1} \text{ is antianalytic} \end{cases}$$

   is called d-holomorphic function[1] on Klein surface.

Atiyah-Weil criterion

1. For given two holomorphic vector bundles say E and F, over Riemann surface say X, a $\mathbb{C}$-linear ringed space morphism,

$$P: E \to F$$

   such that $[P, f]$ is $\mathcal{O}_X$-linear, is called first order differential operator.
2. For a given first order differential operator P, there is an associated symbol map,

$$\sigma_1(P): \Omega^1 \to \mathcal{H}om_{\mathcal{O}_X}(E, F)$$

   here Ω^1 is sheaf of holomorphic 1-forms on X, $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ is sheaf of $\mathcal{O}_X$-linear morphisms.
   Hence we have symbol exact sequence,

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(E, F) \to \mathcal{D}iff_1(E, F) \to TX \otimes_{\mathcal{O}_X} (E, F) \to 0$$

   where, $\mathcal{D}iff_1(E, F)$ is sheaf of first order differential operators and TX is sheaves associated to tangent bundle on X.
3. For a given holomorphic vector bundle say E, symbol map of any first order differential operator can be considered as section of sheaf $TX \otimes \mathcal{E}nd_{\mathcal{O}_X}(E)$. Collection of those first order differential operators whose symbol maps are sections above $TX \otimes id_E$ is called Atiyah algebras, will be denoted by $\mathcal{A}t(E)$.
   For holomorphic vector bundle E, we have Atiyah exact sequence,

$$0 \to \mathcal{E}nd_{\mathcal{O}_X}(E) \to \mathcal{A}t(E) \to TX \to 0$$

   Extension class(obstruction) of Atiyah exact sequence is called Atiyah class $at(E)$, of bundle E.
4. Using čech cohomology, Atiyah[3] proved that $at(E) = -[R^{1,1}]$, also using Chern-Weil theory and Hodge decomposition theorem he proved if holomorphic bundle on compact complex Kähler manifold then all chern classes vanishes and described criterion for existence of holomorphic connection.
   We have used the similar techniques to describe criterion for existence of d-holomorphic connection.

Results and facts

1. Let ∇ be a d-holomorphic connection in d-holomorphic bundle E on Klein surface X_d, then there exists a unique $\mathbb{R}$-linear sheaf morphism $d_\nabla: \omega_d^0(E) \to \omega_d^1(E)$ of degree 1 such that,
   (a) For some open subset $U \subset X_d$, $\alpha \in \omega_d^0(U)$ and $\phi \in \omega_d^0(E)(U)$, $d_\nabla(\alpha \wedge \phi) = (d\alpha) \wedge \phi + \alpha \wedge (d_\nabla \phi)$
   (b) For all $s \in E(U)$ and $Y \in TX_d(U)$, we have $(d_\nabla s)(Y) = \nabla_Y s$.
2. Let E be a d-holomorphic bundle over compact Klein surface $(X_d, \mathfrak{X}_d)$, and let $\tilde{S}$ be the trace pairing $H^{1,1}(X_d, \mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E)) \times H^0(X_d, \mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E)) \to \mathbb{C}$, which is non-degenerate $\mathbb{R}$-bilinear pairing. For $\phi \in H^0(X_d, \mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E))$

$$\tilde{S}(at_d(E), \phi) = \begin{cases} 2\pi\sqrt{-1}\deg(E), & \text{if } \phi = id_E \\ 0 & , \text{ if } \phi \text{ is nilpotent} \end{cases}$$

   where $at_d(E)$ is atiyah class of E and $\deg(E) = \int_{X_d} c_1(E)$, $c_1(E) \in H^2(X_d, L)$ is first chern class of E.
3. For a given indecomposable d-holomorphic bundle E over a compact Klein surface. Then E admits d-holomorphic connection iff $\deg(E) = 0$.

Discussion

1. For given d-holomorphic vector bundles E and F on $(X_d, \mathfrak{X}_d)$, denote by $\mathcal{H}om_{\mathcal{C}}(E, F)$ the collection of maximal families $\{P_{U_i}: E_{U_i} \to F_{U_i}\}$ with the compatibility condition as follows,

$$[P_{U_j}] = \begin{cases} g_{ji}^F[P_{U_i}]g_{ij}^E = (g_{ij}^F)^{-1}[P_{U_i}]g_{ij}^E, \text{if } z_j \circ z_i^{-1} \text{ is analytic} \\ g_{ji}^F\overline{[P_{U_i}]}\overline{g}_{ij}^E = (\overline{g}_{ij}^F)^{-1}\overline{[P_{U_i}]}\overline{g}_{ij}^E, \text{if } z_j \circ z_i^{-1} \text{ is antianalytic} \end{cases}$$

2. Following the case of holomorphic bundle, we have first order differential d-operator with associated symbol map, Atiyah exact sequence whose extension class is called Atiyah class $at_d(E)$, of E.
3. We have Cauchy-Riemann operator $\overline{\partial}$ on Klein surface such that for a d-smooth function $f = \{f_{U_i}\}_{i\in I}$, we have $\overline{\partial f} = \{\frac{\partial f_i}{\partial \overline{z}_i} dz_i\}_{(i\in I)} \in \mathcal{A}_d^{(0,1)}$, also we have Cauchy-Riemann operator for E, $\overline{\partial}_E$ s.t. $(\overline{\partial}_E)_{U_i} = \overline{\partial}_{U_i}$ on some chart (U_i, z_i).
4. For a given d-smooth connection compatible with d-holomorphic structure on E i.e. $d_\nabla^{(0,1)} \equiv \overline{\partial}$ and using čech cohomology, $at_d(E) = -[R^{1,1}]$.
5. On $(X_d, \mathfrak{X}_d)$, we have an orientation line bundle L, Hodge star operator $*: \mathcal{A}^{p,q} \to \mathcal{A}^{1-q,1-p}(L)$ and let $\Phi \in \mathcal{A}^2(L)$ be the volume form given by $\Phi_{U_i} = dz_i \wedge d\overline{z}_i = -2idx_i \wedge dy_i$.
6. Shuguang Wang[6] has described Chern-Weil type result for d-holomorphic bundle and using the fact that top form taking values in orientation bundle can be integrated, we have non-degenerate $\mathbb{R}$-bilinear pair as follows.
7. Let E be a d-holomorphic bundle on $(X_d, \mathfrak{X}_d)$ and E^* be its dual, we have a canonical pairing

$$\langle , \rangle: E \times E^* \to \mathcal{O}_{X_d}^{dh}$$

   which induces $\mathcal{O}_{X_d}^{dh}$-bilinear sheaf morphism

$$\mathcal{A}_d^p(E) \otimes \mathcal{A}_d^q(E^*) \to \mathcal{A}_d^{p+q}$$

   which further induces $\mathbb{R}$-bilinear map,

$$S: H^1(X_d, \Omega_d^1(E)) \times H^0(X_d, E^*) \to \mathbb{C} \qquad ((a, u) \mapsto \int (*\sigma_1)\Phi + \int \sigma_2)$$

   where, $*$ is Hodge star operator and $\sigma_1 + i\sigma_2 = \alpha \wedge u \in \mathcal{A}_d^{1,1} \subset \mathcal{A}_d^2(= \mathcal{A}^2 + i\mathcal{A}^2(L))$.
8. The trace pairing induces an isomorphism $\mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E) \simeq [\mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E)]^*$ and we obtain non-degenerate $\mathbb{R}$-bilinear pairing, $\tilde{S}: H^1(X_d, \Omega_d^1(\mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E)) \times H^0(X_d, \mathcal{E}nd_{\mathcal{O}_{X_d}^{dh}}(E)) \to \mathbb{C}$.

Question

Assuming Hodge decomposition theorem holds true for d-complex manifold(higher dimension analog of Klein surface) of d-Kähler type, using which can we show that if a d-holomorphic bundle on d-complex manifold admits d-holomorphic connection then all Chern classes vanishes?

References

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